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On a Generalization of Magic Graphs

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1. Introduction and definitions

We shall consider a non-directed finite graph $G = [V(G), E(G)]$ without loops and multiple edges and isolated vertices. Let $v_1, v_2, \dots, v_n$ be its vertices and let to each v_i be associated a real number $\rho(v_i)$. If there exists a mapping f from the set of edges $E(G)$ into the real numbers such that

(i) $f(e) > 0$ for all $e \in E(G)$,

(ii) $f(e_i) \neq f(e_j)$ for all $e_i \neq e_j$; $e_i, e_j \in E(G)$,

(iii) $\sum_{e \in E(G)} \eta(v_i, e) \cdot f(e) = \rho(v_i)$ for $i = 1, 2, \dots, n$, where

$$\eta(v_i, e) = \begin{cases} 1 & \text{when the vertex } v_i \text{ and the edge } e \\ & \text{are incident,} \\ 0 & \text{in the opposite case,} \end{cases}$$

then the graph G is called ρ-magic. The mapping f which satisfies the condition (iii) is called a ρ-labelling of G with the indexing vector $\rho = (\rho(v_1), \rho(v_2), \dots, \rho(v_n))$. If f is a ρ-labelling such that $f(e) > 0$ or $f(e) \geq 0$ for all edges of $E(G)$, then it is called a ρ-positive or ρ-nonnegative labelling, respectively. We say that a graph G is ρ-positive or ρ-nonnegative if there exists a ρ-positive or a ρ-nonnegative labelling of its edges.

The special case of ρ-magic graph is the magic graph defined by J. SEDLÁČEK [7] as a graph with a labelling of the edges by positive numbers such that distinct edges have distinct labels and the sum of labels of edges incident to a particular vertex is the same for all vertices. The structure of graphs that admit such a labelling has been investigated from several viewpoints. J. MÜHLBACHER [6], J. SEDLÁČEK [8], B. M. STEWART [9] and M. TRENKLER [11] established some sufficient conditions insuring that a graph is magic. A characterization of regular magic graphs in terms of even circuits was given by M. DOOB [2]. Two different characterizations of magic graphs are given in R. H. JEURISSEN [4], and in S. JEZNÝ, M. TRENKLER [5]. The motivation for the study of ρ-magic graphs is given in [3] of M. DOOB.

We note that the particular case of a ρ-positive graph, if ρ is a stationary vector, has been called in [1] a regularisable graph.

First we shall formulate several necessary definitions.

Under a generalized even circuit D we understand an even circuit C or two odd circuits C_1 and C_2 with one common vertex or two odd circuits C_1 and C_2 without common vertices joined by a path P. A spanning subgraph F of the graph G is called an X-factor of G if none of its components has a generalized even circuit. If an X-factor of G is a ρ-positive graph then each of its edges has unambiguous value. We say that a ρ-positive X-factor F of a ρ-positive graph G separates its edges e_1 and e_2 if at least one of them belongs to F and $f(e_1) \neq f(e_2)$, for some ρ-labelling f of G. As usual $\Gamma(S)$ denotes the vertices adjacent to at least one vertex of S.

2. ρ-positive graphs

In this part we state some results about ρ-positive graphs which we shall use to prove our main result.

Theorem 1. *The following three conditions are equivalent:*
a) A graph G is ρ-positive;
b) Every edge of G belongs to a ρ-positive X-factor;
c) Each connected component G^* of G satisfies: If G^* is non-bipartite, then

$$\sum_{v_i \in S} \rho(v_i) < \sum_{v_j \in \Gamma(S)} \rho(v_j) \text{ for all stable } S \neq \emptyset$$

and if G^* is a bipartite graph with the partition V_1 and V_2 of the vertex set $V(G^*)$, then

$$\sum_{v_i \in V_1} \rho(v_i) = \sum_{v_j \in V_2} \rho(v_j)$$

and

$$\sum_{v_i \in S} \rho(v_i) < \sum_{v_j \in \Gamma(S)} \rho(v_j) \text{ for all stable } S \neq V_1, V_2, \emptyset.$$

Proof. The equivalence of a) and c) is proved in [10] and the equivalence of a) and b) follows from the following Lemmas of this part.

Lemma 1. *If f_1 and f_2 are two ρ-nonnegative labellings of G and α, β two real numbers such that*

$$\frac{\alpha}{\beta} \geq \max\{-\frac{f_2(e)}{f_1(e)} : e \in E(G), f_1(e) > 0\}$$

and $\alpha + \beta = 1$ and $\beta > 0$, then $\alpha f_1 + \beta f_2$ is a ρ-nonnegative labelling of G.

The proof is obvious.

Lemma 2. *Let a ρ-positive graph G contain a generalized even circuit D as a subgraph. Then there exists a ρ-positive spanning subgraph of G which does not contain all edges of D.*

Proof. We consider two cases.

Let D be an even circuit of length s and let f be a ρ-labelling of G and let $m = \min\{f(e) : e \in E(D)\}$. We denote the edges of D by $e_1, e_2, e_3, \ldots, e_s$ in such a way that $f(e_1) = m$. We define a new ρ-labelling h of G:

$$\begin{aligned} h(e_{2i-1}) &= f(e_{2i-1}) - m, \\ h(e_{2i}) &= f(e_{2i}) + m, \text{ for } i = 1, 2, \ldots, \frac{s}{2}, \\ h(e) &= f(e), \text{ for all } e \notin E(D). \end{aligned}$$

By omitting all edges with $h(e) = 0$ from G we obtain a ρ-positive factor F of G which does not contain all edges of the even circuit D.

Let D consist of two circuits C_1, C_2 and a path P or only of two circuits C_1 and C_2, respectively. Let f be a ρ-positive labelling of G and let $m = \min\{m_1, m_2\}$, where

$$m_1 = \min\{f(e) : e \in E(C_1) \cup E(C_2)\} \text{ and}$$

$$m_2 = \frac{1}{2}\min\{f(e) : e \in E(P)\}.$$

We suppose that e' is an edge of D such that $f(e') = m$, if $m = m_1$, or $f(e') = 2m$, if $m = m_2$, respectively. We define an auxiliary labelling q in this way: the edges of C_1 and C_2 have alternating values 1 and -1 and the edges of P have the values 2 and -2 such that the sum of each vertex is zero, and the value of the edge e' is negative, and all other edges of G have value 0. We consider a new ρ-labelling

$$h(e) = f(e) + m \cdot q(e) \text{ for all } e \in E(G).$$

At least one edge e' of the generalized even circuit has value 0 in h. The edges of G having positive value form a ρ-positive spanning subgraph F of G.

Repeatedly using the construction from the proof of Lemma 2 for generalized even circuits we obtain:

Corollary 1. *If G is a ρ-positive graph, then there exists a ρ-positive X-factor F of G.*

Lemma 3. *If G is a ρ-positive graph then every edge e' of G is contained in a ρ-positive X-factor.*

Proof. Let e' be an arbitrary edge of G and let F be some ρ-positive X-factor. If $e' \in E(F)$, then we have nothing to prove; hence suppose that $e' \notin E(F)$. Let f_1 and f_2 be a ρ-positive labelling of G or of the X-factor F, respectively. Using the labelling f_2 we define a ρ-nonnegative labelling f_2' of G in the following way: $f_2'(e) = f_2(e)$ for all $e \in E(F)$ and $f_2'(e) = 0$ for all other edges of $E(G)$.

If $\frac{\alpha}{\beta} = \max\{-\frac{f_1(e)}{f_2(e)} : e \in E(F)\}$ and $\alpha + \beta = 1$, then the labelling $f'' = \alpha f_1 + \beta f_2'$ is a ρ-nonnegative labelling of G and at least one edge of F has value 0. All edges with $f''(e) > 0$ form a ρ-positive spanning subgraph H of G. Let F' be an X-factor of H. If $e' \in E(F')$ then F' is a ρ-positive X-factor of G which contains the edge e' and in the opposite case we repeat the construction described above, since the number of positive edges decreases in every step. By a finite number of repetitions we obtain a ρ-positive X-factor of G which contains the edge e'.

Lemma 4. *If every edge of G belongs to a ρ-positive X-factor, then G is ρ-positive.*

Proof. Let $F_1, F_2, \ldots, F_k$ be a set of ρ-positive X-factors of G with the ρ-labellings $f_1, f_2, \ldots, f_k$ such that every edge of $E(G)$ belongs to at least one of them. The labelling

$$f = \sum_{i=1}^{k} \frac{1}{k} f_i \text{ is a } \rho\text{-positive labelling of } G.$$

3. Characterization of ρ-magic graph

Lemma 5. *If f_1 and f_2 are two distinct ρ-positive labellings of G such that $f_1(e_1) = f_2(e_2)$ holds for two different edges e_1 and e_2, then there exists a ρ-positive labelling f such that $f(e_1) \neq f(e_2)$.*

Proof. We choose two positive numbers α and β such that their sum is 1 and

$$\alpha \cdot \min\{f_1(e) : e \in E(G)\} > \beta \cdot \max\{f_2(e) : e \in E(G)\}.$$

The new labelling $f = \alpha f_1 + \beta f_2$ satisfies the condition of Lemma 5. (Note that if two different edges e_1 and e_2 satisfy $f_1(e_1) \neq f_1(e_2)$ or $f_2(e_1) \neq f_2(e_2)$, then $f(e_1) \neq f(e_2)$.)

Lemma 6. *If e_1 and e_2 are two edges of a ρ-magic graph G, then they are separated by a ρ-positive X-factor.*

Proof. Let f_1 be a ρ-positive labelling of G that satisfies (i), (ii) and (iii), and let F be an X-factor with a ρ-labelling f_2. If F separates the edges e_1 and e_2, then the proof is finished. We must consider only the opposite case. As in the proof of Lemma 3, the labelling of F introduces a ρ-nonnegative labelling f_2' of G. If F does not separate the edges e_1 and e_2, we choose α and β such that $f = \alpha f_1 + \beta f_2'$ be a labelling of G and there exists at least one edge for which $f(e) = 0$. By (ii), $f(e_1) \neq f(e_2)$. Let G' be a ρ-positive factor of g forming by all edges with $f(e) > 0$.

By a finite number of repetitions of the described step of construction we obtain a ρ-positive X-factor separating the edges e_1 and e_2.

The previous Lemmas yield the proof of our main result.

Theorem 2. *A graph G is ρ-magic if and only if G is ρ-positive and every couple of edges e_1, e_2 is separated by a ρ-positive X-factor.*

Note. For the choise of a ρ-positive labelling of G the following lemma may be found useful.

Lemma 7. *The edge e of a ρ-positive graph G has unambiguous value if and only if e does not belong to a generalized even circuit.*

Proof. Let e be an edge of generalized even circuit. We can change its value by an analogous way as in the proof of Lemma 2, when we replace the number m by m' such that $0 < m' < m$. The edge e that does not belong to a generalized even circuit is precisely a bridge connecting some subgraph G_1 to a connected bipartite subgraph G_2 with the partition V_1 and V_2 of the vertex set $V(G_2)$, or an edge added to a connected bipartite subgraph G_3 with the partition V_3 and V_4 of the vertex set $V(G_3)$, respectively. In the first case

$$f(e) = \Big| \sum_{v_i \in V_1} \rho(v_i) - \sum_{v_j \in V_2} \rho(v_j) \Big|$$

and in the second case

$$f(e) = \frac{1}{2} \Big| \sum_{v_i \in V_3} \rho(v_i) - \sum_{v_j \in V_4} \rho(v_j) \Big|.$$

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